THE UNIFICATION OF MATHEMATICS IN THE SCHOOL CURRICULUM.

AT present mathematics occupies an unenviable place in the curriculum; on the one hand, the majority of pupils dislike it probably more than any other subject, and on the other hand, experts in pedagogy seem to have come to a substantial agreement that it is not suited to the abilities of children who are just beginning their school life. The general opinion seems to be that there are difficulties inherent in the subject that prevent the majority of children from deriving anything like the immediate advantage from the study of it that they do from the study of other subjects. Psychologists tell us that the mathematical faculty is developed comparatively late. I believe that these views are all wrong. In the first place, the lives of the great mathematicians show that a true genius for mathematics almost always makes itself evident very early; in fact, earlier than any other intellectual faculty (Gauss discovered an error in his father's accounts when he was three years old); but, of course, the education of geniuses is not the main object of the schools. In the second place, I believe that the want of effectiveness of mathematical studies is chiefly due to a bad arrangement of the curriculum and to false notions concerning the immediate objects of such studies—perhaps to the absence of any immediate object. With a proper arrangement of the work, mathematics ought to be as effective and as immediately useful as any subject taught in the schools. On numerous and various occasions during the last ten years I have advocated the unification of the elementary mathematics, and I gladly take this opportunity to explain what I understand by such unification.

It is not always immaterial in what form a question is discussed, and this is particularly true of the matter in hand. The reformation of the mathematical curriculum of the secondary schools alone will not answer our purpose; we must simultaneously reform that of the lower grades, upon which that of the higher schools

must be based. Therefore I shall consider the ground that ought to be covered, as it seems to me, in the primary, grammar, and secondary schools together. I believe that all instruction, to be effective, must have an immediate object—an object attainable by practically every pupil to whom it is given; mathematics has no place in a liberal education unless it is taught with such an object. It seems to me that, in a certain sense, the schools are trying to do too much in teaching many things that are purely technical, on the one hand, and many that are adapted only to pupils of distinctly mathematical tastes, on the other hand. I suppose I shall shock many of you by saying that it is my very decided opinion that the study of Euclid, otherwise "elementary" geometry—I prefer to call it "demonstrational" geometry—is, as such, a waste of time for all but a very small proportion of those who are now required to pursue it; not that I would omit it altogether, as you will see later, but that it should be made subordinate to a more useful kind of geometry, which may be called "observational" geometry, the geometry of fact. There was a time when the so-called geometrical method was the standard scientific method of treatment for all mathematical problems; but that time has passed, and the spirit of modern mathematics is, not geometry, but algebra, or, to go back to the first principles, arithmetic. Euclid is, to be sure, one of the mathematical classics, but, as a system of methods, it is, like many other classics, archaic. You will probably more nearly agree with me in the belief that mercantile arithmetic should be reduced to a minimum and regarded only as a series of applications of general principles in the schools of liberal learning, to be mastered more thoroughly, if needed for practical purposes, in the commercial college or the world of business, as any other kind of professional technique is acquired.

Mathematics is often distinguished from the natural sciences by calling it the deductive science, while they are called sciences of observation; in reality, mathematics is itself a science of observation—its concepts are the immediate products of observation, from which also its fundamental principles have been derived by induction. The great difference between it and the other sciences

is that in it induction has already led to a sufficiently complete system of fundamental principles to admit of systematic deduction from them, whereas the other sciences (physics and astronomy to a much less degree than the rest) are still in the rudimentary stage in which the processes of observation and induction have not yet been carried so far as to make deduction possible to any great extent. In this sense alone is mathematics different in its nature from the other sciences. This more perfect character of mathematics renders it the best possible means of developing the power of observation, for here alone can the results of induction from observation be verified with certainty, and here alone can the train of thought started by nature (that is, the physical universe)—namely, observation, induction, and deduction—be completely carried through. Moreover, the education of the individual differs from the life-history of the race just in this, that the pupil is made to pass through the essential stages of development without wasting his time on what the experience of former generations has shown to be unessential. In other words, education is selective history, and whatever mode of selection most thoroughly excludes the unessential is most economical, enables the pupil to master the largest amount of what is essential, and gives him most time to devote to the exploration of new fields when he shall have explored the old; that is, leads most rapidly to independent thought, the true goal of education. Furthermore, deduction reacts upon the earlier processes beneficially in that the conditions necessary for its prosecution suggest, not only the principles upon which it must itself be based, but also the elementary facts from which these principles are to be derived by induction. Thus it has come about that the facts to be determined by direct observation are very much more elementary in mathematics than in any other science, and are, therefore, the more readily observed and fixed in the mind. Unfortunately for the practice of education in general and for the popularity of mathematics in particular, the observational method of teaching our subject has been neglected, has never been suspected to be possible by some educators, and is considered by others to be altogether inappropriate. It is

perhaps unnecessary to say that a considerable part of the work of the kindergarten is the most elementary kind of mathematics; the pity is that this work is not sufficiently systematic, consecutive, and progressive, and is not continued along natural lines in the primary schools. I think that what I have said justifies the claim that mathematics is the ideal study for the inculcation of the habit of observation in the earliest stages of school life.

It seems to me that the immediate object of education can be best described as the formation of good habits, and that of liberal education as the formation of good habits of thought and expression; the acquisition of knowledge is the storage of food for thought, and, therefore, is a secondary object. The chief strictly mental faculties whose use is to be made habitual are attention, observation (which is not by any means the same as attention), concentration, and reason—just the faculties that are called into play particularly by the proper study of mathematics. But these faculties cannot be developed by precept; it is by practice alone that habits are formed, and it seems probable that they are most readily formed when the child is unconscious of outside influence; this would suggest that the child is to be led and not driven, and the surest and easiest way of leading him is by rousing his interest. The apathy of the teacher, his own want of interest, and his failure to arouse the interest of his pupils (which may require heroic effort on his part) are perhaps the most potent factors in the relative inefficiency of the schools in teaching mathematics as compared with other subjects. The child naturally takes most interest in that with which he comes into closest contact, in those studies of which he sees the most immediate, the most frequent, and the most profitable applications. The remedy for the present unpopularity of mathematics is to make it practical, to make it immediately useful, and to exclude from it everything that is not directly applicable to the child's world, certainly in the early stages.

I have spoken of mathematics as a science, but this is only one phase of it, and by no means the most important. The science of mathematics is only a means to an end. Mathematics is really an art; it has for its object the accomplishment of

something useful in itself; it is not a mere system of mental gymnastics. Mental training is, to be sure, of prime importance in any system of education, general or special—physical, manual, or intellectual. The athlete or the mechanic can do first-rate work only when his mind has been trained to think quickly and correctly. But mental training of a useful kind comes from practice in doing something, in accomplishing what is itself useful (or may be so on occasion), in overcoming difficulties, in acquiring good habits, especially in doing what is distasteful, if it be proper, or charitable, or agreeable to others. The habit of doing what ought to be done and the acquisition of the power of doing something well should be the chief aims of every self-respecting youth, and the schools should be so managed that these objects may be accomplished. The former result can be enforced only by a fixed curriculum, while the latter is often facilitated by elective studies, although it may well be attained without electives, if the curriculum is sufficiently broad. In this connection I wish to say that I have no sympathy with the notion that a child should never be required to do what he doesn't want to do. I often do what I don't want to do, and I regard it as one of my best powers that I can compel myself so to do. Unless the child acquires this power early, he is not likely ever to possess it in any high degree; when acquired, it is so useful that it may almost be said to cause one to wish to do what he knows is very distasteful to him. How much control of one's self such a power gives is evident.

I take the ground that the object of mathematical instruction in the schools is the development of power, of the ability to do something, and not merely the acquisition of knowledge regardless of its use. Knowledge is power just in so far as it enables its possessor to do what he could not do without it. I believe there is no kind of knowledge that cannot be made useful in action, and I claim that all knowledge should be imparted in such manner that this use can be made of it. Unfortunately, the greater part of the mathematics of the schools is popularly regarded as quite useless to the great majority of pupils excepting as a mental discipline; the bad name thus given to it (one

can hardly help saying deservedly, in the present state of things) is not due to any peculiarity of the subject, but simply and solely to the manner in which it is commonly taught—particularly to the want of purpose in such teaching.

Mathematics is an art; like other arts, it is characterized by its methods; as in the case of other arts, there are geniuses to whom it is a second nature—who absorb it through their pores, so to speak, with little or no assistance from others; doubtless there are some few who cannot grasp it at all, but the great majority of people can, if properly taught, acquire considerable facility in its practice, certainly sufficient for the purposes for which it is most commonly used. I speak now only of the elementary mathematics, the mathematics of the schools; I am far from thinking it advisable for those who have no special predilection for it to study the higher mathematics. Like other arts, also, mathematics has its technique, its system of standard modes of procedure derived from the experience of the great mathematicians, which must be thoroughly mastered before independent work can be undertaken with any hope of success; and this mastery requires years of hard work even for capable students. The mathematics of the schools, and practically all that is taught in the college, belongs to the technique of the art. Technique can be learned only by practice, and cannot be properly understood until it has become a habit, to be applied almost mechanically, without more thought than is necessary to keep the attention fixed on the matter in hand, without hesitation, and with absolute accuracy. Whoever is to know anything of mathematics must begin with the technique, and he who does not aim to become an expert mathematician will confine his attention to the technique. The mere computer, however skilful, however phenomenal, is only an unconscious practitioner of the technique, a skilled artisan, but not an artist.

The fact that what the child learns, or can be expected to learn, of mathematics in the schools is only a part of the technique of the subject has a direct bearing upon the methods that ought to be employed in teaching it. The teacher and the pupil have their separate rôles; neither can do the work of the

other. The teacher must take the initiative and show the pupil what he is to do and, by example more than by precept, how it is to be done; but, when he has acted his own proper part, he must still be the constant director and, if need be, prompter of the pupil, laying out his work for him and seeing that he does it. The attitude of the pupil should be that of an apprentice learning a trade; he has first to find out what is to be done and how he is to do it, and then he must practice it until he shall have acquired the proficiency necessary to do it skilfully, rapidly, and accurately. The question of why it is done in the manner shown ought not to be raised, at least not until the pupil has had considerable practice in the method and ample time to think about it. Reason is the highest faculty of the human mind and the slowest in its development. Children may ask questions, they are very inquisitive, but they do not naturally reason logically to any great extent, nor can they truly appreciate the force of explanations of whys and wherefores that may be given them, excepting in the simplest cases; and it is requiring too much of them to expect them to understand why certain processes are carried out in certain ways—it is as if you should expect them to understand why a word is spelled in a certain way; the fact that they are so carried out ought to be enough, and the ability to carry them out so is quite enough. Instead of explaining the reasons for the rules of arithmetic, for example, the teacher would do better to explain the meanings of some of the so-called practical problems that are to be solved by application of the rules. Professor Klein classifies mathematical mind as of three types, often more or less combined in the same individual—intuitive, formal, and critical; the critical type is sufficiently common to make it pretty certain that no really incorrect rule will find its way into the generally recognized system of technique, nor, being there, will long remain. Unfortunately, some of the text-books often used in teaching certain branches are not altogether free from the errors of former generations, for it is only recently that the ideas of modern mathematics have exerted any considerable influence on the schools in this country. But it requires the most untiring patience and ingenuity of the formal mathematician to develop

any branch logically, and the most strenuous efforts of the critic to make sure that the development is logical. No branch of mathematical technique, as expounded in the current text-books, not even much-lauded geometry, has a strictly logical foundation; every one of these branches is based upon fact as the mathematical world knows it by experience. To teach arithmetic upon a strictly logical basis would require the explicit statement of several hundred distinct propositions, and would impose a strain upon the mind of the pupil that no child could endure. If, then, we cannot teach mathematical technique logically, we ought not to make a pretense of doing so. When the pupil has become somewhat familiar with methods and with what may be called the fundamental facts of a branch of mathematics, he may be encouraged to analyze the methods and to determine the particular facts upon which they are based. Mathematics has been regarded as a deductive science, but all deduction must be preceded by analysis and induction, by which alone the fundamental principles can be even hypothetically determined.

The classification and formalization of mathematics has been only a sequel to the discovery of isolated facts, and was impossible until such facts were known. It is, likewise, impossible for the child to learn to reason logically until he has acquired a body of knowledge, and, in the ordinary course of events, his reasoning power will grow stronger and more acute as his knowledge increases. The chief function of the teacher in this work is to provide the pupil with food for thought graduated to his power of assimilation and, what is quite as important, connected and always progressive. Too much emphasis cannot be laid upon the importance of progress in mathematical exercises; if the pupil is not learning something new to him, he is probably losing control of what he had possessed. In mathematics, the opportunities for applying the principles involved in earlier work are so frequent that I am inclined to think it better, when a child does not attain the desired proficiency in one topic after what appears to be a reasonable length of time spent on it, to pass on to the next topic, in the hope that he may master the earlier principles gradually as he becomes more familiar with them, than to

compel him to stick to one thing until, as is likely to happen, he becomes disgusted with it or, what is just as bad, becomes apathetic.

While I disapprove of requiring or even encouraging pupils to discuss the reasons for mathematical methods before they are pretty familiar with them, there is one point that I regard as very important, although it is often neglected, namely, precise expression. The pupil should be required at every step, regardless of the time it may take, to state his problems, rules, processes, and results—indeed, everything that is connected with his work—in exact and grammatical language, preferably in his own words. It is not enough for the teacher to see that he understands a point; he must be required to state it, and his statements must be carefully scrutinized and openly criticised. The fearful prevalence among school children of the habit of using slang is, it seems to me, largely due to neglect on the part of the teacher to insist on the use of good grammatical English on all occasions. If children are not corrected whenever their language is bad, how can they be expected to acquire facility in the use of good language?

We are often led astray by mere words, especially in discussions between conservatives and radicals whose points of view are very different. I say that all instruction should be logical, and I do not mean that it should consist of specimens of formal logic presented as such. I regard explicitly logical and philosophical studies as out of place in the school curriculum. But the child must learn to think properly; he must get into the habit of drawing exact—that is, logical—conclusions. He cannot be taught this by precept; he will acquire it but very slowly and uncertainly, if left to himself. He must be guided in the right way, by example, by the constant exhibition of good forms, and the persistent criticism and correction of bad forms, whether his own or those of others. He must be shamed out of practicing what he knows is wrong. I think teachers are too apt to confine their efforts to the immediate requirements of the subject they happen to be teaching at the moment. The practice of the principles of gammar, of logic, and even of rhetoric must

be made compulsory in connection with all school work, although these principles may be formally unknown to the child. By such practice alone will the child learn the principles on which his future studies in these subjects as such must be based. It may seem strange that the logical habit is more certainly acquired to a considerable extent than the grammatical habit; but the former is recognized as advantageous, and even necessary, as the natural mode of determining actions, while the latter is regarded as a mere accomplishment of no particular utility to those who are to spend their lives in action.

The most essential element of success in school work is the personal efficiency of the teacher, and I cannot but think that it would be a great step forward to replace the text-books in elementary mathematics by collections of graduated exercises, leaving it to the teacher to give directions for the solution of the problems in his own way. Certainly, a pupil should never be expected to learn fundamental principles or methods from a text-book.

The first question to be answered in laying out a curriculum in mathematics is: At what stage of mental development—or, if you please, at what age—should the pupil begin the study of mathematics? I believe it is best to begin instruction in any necessary subject before the pupil has acquired of himself any appreciable knowledge of it; perhaps the best time would be when he is just beginning to think about it, if that time could be determined with any certainty. Such a course would save both teacher and pupil much trouble in eradicating wrong ideas that had been formed. This remark has particular reference to the fact that children generally come to the study of geometry with preconceived notions that are very detrimental to a proper appreciation of the importance of the subject as taught and its value as a branch of knowledge. The study of algebra as a distinct subject is objectionable in that the true significance of the subject as generalized arithmetic is thereby concealed, and in that the pupil is thus led to think that he has finished it when he has learned all that the text-book contains, whereas the subject practically includes the whole of modern pure mathematics. It seems to

me very important that the first ideas formed should be correct, and I do not see any greater difficulty in teaching the elements of mathematics to very young children than in teaching them the elements of drawing; indeed, these two subjects are very closely related, and might be taught together to some extent with advantage to both. The earliest instruction in mathematics, in my scheme, would be imparted by what might be called the kindergarten method, with a very gradual transition, as the mental development of the pupil progresses, into the more strictly scientific method. If I had to assign names to characterize the subject-matter, I should call that of the first two years, more or less, number and form, although I do not consider it wise to let the pupil know that he is learning any particular branch of mathematics; in fact, I propose to arrange the course in such a manner that it would be very difficult to determine just where one branch ends and another begins.

The curriculum that I propose is founded upon certain principles that seem to me to be essential to proper instruction in mathematics as a subject to be continued to the end of the secondary-school course. It is advisable to bear in mind that the mathematics of the schools is only a very small part of what is now known of the subject—the most fundamental part, certainly, and the most generally useful—and is, to some of our pupils, a preparation for still further work along similar lines. The description of the curriculum will be facilitated by a statement of the principles on which it is based.

1. Elementary mathematics is to be regarded as one subject. This is the keynote of my curriculum.

Neglect of this principle leads to false ideas of the nature of mathematics and of the extent of its various branches. I suppose mathematics is, to most school children, only a name used to designate arithmetic, algebra, and geometry, taken collectively; they see nothing common to these branches in subject-matter and methods, and get from them no notion of what the higher branches may be. They find that it requires a distinct effort to become familiar with the concepts of each new branch, and are often restrained from pursuing mathematical studies

farther than they are obliged to by a disinclination to make this effort. Moreover, they imagine that what they learn of each branch in school is practically all that is known of it, and thus put a wrong estimate on their attainments. I should even avoid the employment of technical names for the different branches, although such names are useful when applied to methods; certainly, the early work in arithmetic and geometry might be conviently designated as number and form, if names for them are thought to be necessary.

2. Mathematical instruction should be always connected, consequent, and progressive.

There should be no period of school life during which mathematics is not studied in some form. How much the pupil loses by discontinuity of work must be evident to every teacher immediately after the summer vacation, and yet there is a period of at least one year in the curriculum of most secondary schools in which strictly mathematical studies have no place. It is better to devote less time to the subject in some grades than to omit it entirely from any grade.

Each topic should be the immediate outgrowth of previous topics and intimately related to the preceding topic or, at least, to a topic very recently considered. In a curriculum so arranged no essential change in subject-matter or method will be perceptible to the pupil at any stage; he will not be aware of any distinct effort to assimilate new ideas, but each step will be to him a natural continuation of his previous course. Thus, the algebraical notation should be introduced at the earliest possible point in the study of arithmetic, certainly as soon as the child has acquired some fluency in the fundamental operations, and should be thereafter employed on all suitable occasions; there will then be an easy and natural transition from discrete number to algebraic expressions and equations. Algebra is, in fact, only a generalization of arithmetic involving an extension of the idea of number —"universal arithmetic," as Newton called it. This course will render it impossible to draw any hard and fast line between arithmetic and algebra—a desirable result, as I indicated in connection with the former principle. Again, demonstrational

geometry grows naturally out of observational geometry, in which simple demonstrations based largely upon the method of superposition will have suggested themselves more and more frequently as the work progresses.

There should be no time at which the pupil is not learning something new to him. Reviews should always take the form of new applications of old principles, for which there will be sufficient opportunity at any stage. Mere repetitions tend to weaken the child's interest in his work and to harden the mind to new impressions.

3. The work of each stage should be adapted to the child's mental development.

In the earlier stages observational and constructive methods should be employed exclusively. Terms should be defined only as they are needed for actual use, and never in anticipation of future needs, very few at a time, so connected that no one will be overshadowed by others, in words familiar to the pupil, and each definition should be accompanied by numerous and typical illustrations. Whenever it is possible, the types should be exhibited, and the pupil should be required to state in his own words what they seem to him to have in common; that is, the pupil should be made to frame for himself a definition derived from actual observation of typical cases. This is the historical method, and in its application the function of the teacher is to present suitable types in such variety that the child will readily eliminate irrelevant elements; in other words, it is the teacher's duty to abbreviate history for his pupils, in order that they may learn its essential results without being compelled to live the whole life of the past. Merely formal definitions without illustrations that the child can comprehend should be entirely excluded, but exactness of expression should be insisted upon in such definitions as are admissible and in all other statements.

4. Exact expression is a *sine qua non* of satisfactory work in mathematics.

I believe that mathematical studies are better adapted than any others to inculcate the habit of exact and correct expression, and that this is one of their most important functions as ele-

ments of a liberal education; in order to secure this habit, however, the teacher must subject inaccurate expression to constant and open criticism. The pupil should be required to state in his own words, but in idiomatic English and with such regard to style as corresponds to the knowledge of the vernacular that he may be reasonably expected to possess, all facts brought to his attention, the general principles he derives from the facts, and the conclusions he draws from these principles. It is a very common and grievous error for a teacher to say to a pupil, "Oh, I see that you understand that point; we'll pass on to something else," when the child has not so expressed his thought that there can be any certainty of its correctness. I feel that it would be impossible to insist too strongly on this point. But the ultimate object of such insistence is to make correct expression spontaneous, and, therefore, memorizing should be explicitly discouraged in mathematical studies, except in cases where retention in the mind is the main object. I know of nothing in elementary mathematics that need be committed to memory, excepting the arithmetical tables of addition, multiplication, weights, measures, and values, and a few formulæ of algebra and trigonometry. The propositions of geometry should be thoroughly understood, so that they can be expressed by the pupil in his own words, but they should not be memorized in set words nor ever quoted by number.

5. Every step taken should have a definite, evident, and useful object.

This principle is in accordance with the conception of mathematics as an art. Every application, however, must be to things with which the pupil is familiar and in which he takes a present personal interest. One of the great problems for the teacher to solve is how to excite an interest in matters to which the principles being taught are applicable. There should never be the least doubt whether the meaning of a problem proposed for solution is clear to the child's mind or not, and problems whose solution are of no practical value in themselves are to be systematically avoided. Any particular class of pupils should be treated according to the probable future requirements of its

members; but as a rule, technical questions should be made subordinate to general methods. In my opinion, no subject should be taught solely, or even primarily, as a means of mental discipline; I think that any kind of mental training that is valuable can be affected by forms of activity whose immediate products are useful; and I doubt whether real mental strength is ever developed by mental gymnastics. I think that some branches of mathematics have suffered very much from being regarded as valuable chiefly for their disciplinary effects.

6. Any topic about which the child is liable to pick up false notions if left to himself should be taught before such erroneous ideas have had a chance to fix themselves in the mind.

This applies to the use of words as well as to facts and methods. A good deal is said nowadays about the danger of too early instruction in certain subjects; for example, it is said that the child should not begin the study of arithmetic before his tenth year; and when I ask if he shall not be taught earlier to count, I am told that he ought not to receive systematic instruction in mathematics before that age. Now, I do not know what is meant by unsystematic instruction; it seems to me that all teaching should be what I call systematic; I do not see how absence of system in instruction can make any subject easier, nor do I think that a method that makes a subject unnecessarily difficult is suitable for any age. If the pedagogical objection is really to pure theory, I say simply that I do not believe metaphysics has any place in the school curriculum. Again, the child learns to best advantage when he is unconscious that he is being taught, and no mode of teaching that excites self-consciousness in the pupil is really good; but it is hardly possible to implant in the child's mind ideas that are entirely new to him without making him conscious of outside influence and of the fact that he is acquiring notions that did not originate with him; such instruction should be kept within the narrowest possible limits, and certainly not extend beyond the most fundamental elements, from which he can and should build up his whole mental world for himself under proper guidance and without apparent compulsion. The kind of self-consciousness that is injurious to the

child is the recognition of his organs, his mind as well as his stomach, as something having an individuality apart from his conscious being, his true "ego." But the belief that he is evolving what he learns out of his own mind is equivalent merely to the recognition of it as a logical consequence of certain premises, and is the basis of exact thought, the best product of education.

I should, then, begin instruction in number and in form at the same time—that is, at the very beginning of school life—and continue them throughout the course simultaneously, apportioning the whole time available for mathematical study in each grade between them in such a way that, at any stage, the pupil shall have acquired the knowledge of either that may be necessary for further progress in the other. If the work is so distributed, the child will never come to an application of number to form nor of form to number until he is prepared for it. The work along each of these lines must be logically consecutive and progressive, and should be connected as closely as possible with the work along the other line—if not with what is being done at the time, at least with previous work—so that the two shall be inseparably woven together in the child's mind. I think we ordinarily pay far too little attention to the application of number to geometry; in fact, under the prevalent conditions, the child gets hardly any practical knowledge of metrical geometry, of which the greatest use can be made in the actual affairs of life. I would make the study of mathematics consist always in doing something; I would make the theoretical part of geometry entirely subservient to constructions and calculations. The child should learn to use the ruler and dividers as early as possible, and neatness and accuracy of drawing—to a definite scale, whenever practicable—should be regarded as important elements of good work in geometry and its applications. By far the most important method of proof in geometry is the method of superposition, and any child can be made to understand it if it is presented in the right way, that is, in the concrete form. The great difficulty in it, at present, is that one figure is not actually applied to another, but the application is only imagined, and it is the want of geometrical

imagination that prevents many children from comprehending geometry. In the earliest stages the figures should be concrete, cut out of paper or other suitable material; and one actually applied to the other. In the course of time, after a sufficient number of actual applications have been made, the pupil may be exercised in imagining the applications, and, if his imagination is at fault, his errors may be corrected by actual applications. With practice under proper guidance all difficulties will disappear. But now the pupil has no time for such practice; he is rushed through geometry in a year, held to the demonstration of so many theorems a day, without regard to his preparation for the work and his comprehension of it. And what is it all for? The chief object seems to be that he may say that he has studied geometry, because the college requires him to be able to say that. Nevertheless, the right kind of geometry is of practical use and of daily application in some form or another, and it is this kind that ought to be taught in the schools. If anyone who had learned geometry in this way were sufficiently interested to care to read a formal presentation of the subject, he would find no difficulty in Euclid; but such reading is appropriate to philosophers and not to school children. It is evident to the child that he is accomplishing something when he is making a geometrical construction, and such constructions are just the things on which he gets the firmest grasp, and which he retains to a certain extent after geometry as a system has faded out of his mind. Such construction should, then, constitute a very considerable part of the work in form; the greater the care bestowed upon them and the more the pains taken with them, the more permanent is the effect of such exercises and the better are the things they represent understood. The metrical principles—which, in their elementary forms, are all derived by the method of superposition—should be constantly applied to practical problems, to the actual determination of lengths, areas, and volumes, in terms of various units. I have found that many boys who seem to have no difficulty in proving and remembering the formulæ for measurements are completely at a loss how to proceed when they are asked to apply them to concrete cases. This is not surprising, as nearly

all their practice in measurement is had in connection with the study of arithmetic at a time when their knowledge of geometry is entirely insufficient, whereas they have almost no practice in connection with the corresponding work in geometry. But it is not enough that these principles should be applied to the geometrical forms as such; they should be applied also to concrete objects whose shapes are sufficiently simple and to such as have forms approximately compounded of simple forms. In practical life, approximate measurement is quite as important as accurate measurement, and is often alone practicable; attention ought, then, to be paid it in the schools. As I have indicated, I would make practical use the basis of all instruction in mathematics. I do not mean that I would teach the child to do only those things that he may want to do in after-life, but that I would teach him to do as many of those things as possible as parts of a systematic course in mathematics.

What I have just said about the work in form is to be applied, as far as it is applicable, to the work in number also. All the principles learned should be applied to concrete cases within the child's actual sphere of interest, and this sphere must be so extended, if necessary, as to embrace a large number of applications of each principle. This is where instruction in arithmetic is often faulty, in that the teacher does not secure a sufficient extension of the child's sphere to include the applications made; such extension, if necessary, should be regarded as a preliminary requisite to each application. The comprehension of the conditions imposed by the statement of a problem, the recognition of what is required, and the perception of the relation of the required thing to the things that are given, are just as important as the knowledge of the process by which the required thing is to be found and the ability to carry this process out. This is something altogether different from the knowledge of reasons for the rules in accordance with which the fundamental processes are performed. The one is sense, perception; the other is meditation, philosophy. The best possible basis for a curriculum in elementary mathematics would be a series of problems in number and form whose solution should require a knowledge of all the

principles to be taught, arranged according to the proper order of the corresponding principles. These problems should be taken up consecutively, and each solved after the conditions necessary for its proper solution have been secured. The solution of other problems involving the same principles will furnish ample practice in the applications of these principles. In this connection, I am reminded of a Harvard professor who, when criticised for setting the same examination paper that he had set ten years before, replied that he would be perfectly satisfied with any student who could answer all the questions he had set in the last ten years. So I should be satisfied with the mathematical knowledge of any child who could solve all the problems in such a standard series. At all events, the solution of useful problems ought to be the main thing, and everything else should be subordinated to that; theory should be learned by the practice of methods in concrete cases.

In what is recognized as one branch, I suppose nobody doubts that the transition from one topic to another should be as gradual and as natural as the circumstances of the case allow. But the same is true also of the transition from one branch to another, in so far as the latter branch can be regarded as an outgrowth of the former. This applies particularly to algebra as an outgrowth of arithmetic and to demonstrational geometry as an outgrowth of observational geometry. Algebra is in reality nothing more than symbolic arithmetic; it is, in its elementary part, a mode of representing arithmetical processes and relations; its higher parts deal with the generalization of these processes and relations. Its symbolism makes it a more powerful instrument for the solution of numerical problems than common arithmetic, in that the results obtained by it are not restricted by particular numerical values of the given quantities. In arithmetic (and also in geometry) particular cases are discussed; the results would be valueless for general purposes if it were not that each particular case is considered as a typical representative of a class of cases to all of which the same discussion is appropriate, step for step, and that this discussion leads to analogous results in all cases of the same class. The difficulty is that the individual is left to his own

resources to determine, by unconscious inductive reasoning, whether the methods employed in the discussion of one particular case are or are not applicable to any other particular case that may arise. In algebra, on the other hand, a whole class of cases is discussed at once under such preliminary conditions as make it evident what particular cases belong to this class. In other words, the method of arithmetic is inductive, while that of algebra is deductive; but the deduction of algebra is based upon the induction of arithmetic, of which it is the immediate outgrowth. The early part of algebra consists simply in translating the language of arithmetic into symbolic language and *vice versa*; the earlier this translation is begun, the more readily will the child learn to think in the new language, and the easier will he find the study of algebra as such when he comes to it, because he will then have surmounted all the obstacles due to the mode of expression and will have to contend only with those difficulties that are peculiar to the generalization of number. He ought, therefore, to learn the language of algebra while he is still studying arithmetic—in fact, as early in the study of arithmetic as possible—and use this language on all suitable occasions, in order that he may have acquired fluency in it before he begins the study of extended number. This is in accordance with the pedagogical principle that difficulties should be overcome separately, as far as it is practicable. I do not mean that the child should be taught merely the notation of algebra, but that he should be exercised continually in carrying out the same lines of thought in the symbolic language of algebra that he would have to carry out in arithmetic, by which means he acquires unconsciously the fundamental notion of an equation and of the transformations to which an equation may be subjected. Due regard being had to the results derived by induction from the discussion of any particular problem by the purely arithmetical method, this method is as general as the algebraical method, although this generality is not so evident. The reason why some numerical problems are more readily solved by algebra than by pure arithmetic is that the symbolic notation enables us to carry out operations involving any numbers even before those numbers are

known. The objection that appears to me most likely to be raised to this early introduction of the algebraical method is that it encourages the child to use this method when he ought to be having practice in arithmetic; but I take the ground that the study of arithmetic is justified only in so far as it enables the pupil to solve problems, and that any problem ought to be solved by algebra if it can be better solved in that way than by arithmetic. Practice in the fundamental operations is the most important thing in arithmetic, and this practice can be had just as well in connection with algebraical work as without it.

What I have said about induction in arithmetic does not apply to the very first elements of the subject. The study of the fundamental operations—that is, of the relations of numbers to each other—has three stages. First, all the mutual relations of the very smallest numbers must be learned by actual observation and fixed in the mind; this stage ends with the commitment to memory of the tables of addition and subtraction (the latter table is only another mode of reading the former and need not be learned separately), which assures the immediate command of the results of observation. Second, the multiplication table must be thoroughly memorized; this is not a collection of the results of the child's personal observation; it would not be economical to require the child to verify all the numbers of this table, nor would such a verification answer any useful purpose; the mastery of the table is a necessary condition for further progress, but the table itself must be taken for granted as the result of the experience of others. The third stage is the study of the mutual relations of numbers in general based upon the tables of addition and multiplication of small numbers; our system of numeration, of which the elements will have been learned in the first stage, and whose indefinite extension constitutes the beginning of the third stage enables us to make this generalization. As I have said, the fundamental operations constitute the most important part of arithmetic as a subject of the curriculum; they should be practiced separately until they can be performed accurately and rapidly, in useful applications, as far as may be possible. It is astonishing how few people can add, subtract,

multiply, and divide with accuracy, to say nothing of rapidity, when we consider how much time is devoted to arithmetic in the schools. I was informed by an astronomer some years ago that it was almost impossible to find young men pursuing their studies in college on whose computations he could place any reliance. I attribute this state of things to the fact that the fundamental operations are neglected in favor of technical and especially mercantile applications, whose intricacies distract attention from the operations themselves. Algebraic symbols should be introduced as soon as the fundamental operations have been mastered. As to fractions, it seems to me simplest to treat decimal fractions by the rules for integers before the general principles of vulgar fractions are introduced; I take it for granted that the fractions whose terms are very small integers will have been studied in connection with those terms. The rest of arithmetic ought to be regarded as applications of the fundamental operations, percentage as an application of decimal fractions, and interest of percentage; mercantile applications should be reduced to the minimum and should involve only such transactions as the child can understand without much preparation; square root is best taught by the algebraical method, and cube root should be so taught, if at all. One of the great stumbling-blocks in arithmetic is denominate numbers; they should be taught only in the forms in which they will be actually used; many of them are mere curiosities not likely to be used by one out of a thousand. What I mean by "the forms in which they will be used" is illustrated by the table of long measure, which contains several units not now commonly employed and several others that are practically never used together. Especially should improbable applications be excluded. When the metric system of weights and measures shall have been generally adopted, the difficulties of denominate numbers will have disappeared, excepting in so far as they arise from the English monetary system.

When the purely arithmetical work is finished, algebra as such ought to be continued—not begun again, but continued from the point reached in connection with arithmetic. By this time the simple equation of the first degree in one unknown quantity will

be a familiar idea. The next step is naturally to the conception of negative number (I should say this also ought to come in arithmetic), and the development of algebra from this point will proceed in the usual way.

In form I should begin by teaching the pupil the names of the simpler plane figures and solids, putting into his hands models of a considerable variety of each form, naming their parts (vertices, sides, edges, and faces), making him count these parts (as an exercise in number, closely following his progress in that line). The child seems to have a veritable mania for names, coining words of his own and using those coined by other children, of which many have no meanings at all and yet are used in certain definite connections, so that there ought not to be any difficulty in teaching the names of forms. There seems to me to be no better concrete representative of a number than a geometrical form; I believe that the simultaneous study of number and form, while it gives an additional interest to the latter study, will greatly facilitate the assimilation of the former. The next topics are naturally the applications of figures to each other, measurement (of lengths, areas, volumes, and angles, in terms of various units, first arbitrary units and then those in common use), the use of the ruler and dividers, parallel lines (without regard to the theory), and similar figures. Of course, care must be taken to distribute the time between number and form in such proportions that progress in the one shall keep pace with that in the other. During this stage a number of the simpler geometrical theorems can be proved, but no systematic attempt to prove theorems should be made; the proofs given should grow naturally out of observation and construction; the child must be led gradually to see in what a proof consists and he will then devise proofs of himself in many cases. Then all the principles of measurement can be taught in their practical applications, and I should lay great stress on this part of the work; such measurements constitute the most important uses of geometry.

Finally, with this preparation, there is no real difficulty in teaching plane trigonometry as the cap-stone of elementary mathematics. From the time when the use of the ruler and

dividers has been learned, constructions with them should constitute a regular part of the work, each figure studied being accurately drawn by the pupil to a definite scale. When the child has grasped the nature of a proof, he should be encouraged to prove every result he obtains by inspection or otherwise and to state his reasons for the constructions made; the teacher should cause him to correct his errors by questioning and criticising; if such methods fail with a particular pupil, it is better to call on other members of the class to remove the difficulty than for the teacher to correct the error himself.

Such are the main features of the curriculum in mathematics that seems to me most suitable for the schools, most readily made effective, and most useful to the pupil when his course is finished. Perhaps it demands more effort of the teacher than the present curriculum, for I assume that all instruction will be given in the class-room, leaving only exercises to be done outside; but it will save time, and I am confident that the results will justify the effort even to the teacher.

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[This number is, in accordance with our custom, made up of papers read at the annual meeting of the Association of Colleges and Preparatory Schools of the New England States. It had been intended to include more than those published here, but unforeseen delays have prevented the authors from forwarding the manuscript in time for this number.—THE EDITOR.]